Inter (Part-II) 2018

Mathematics	Group-I	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: (16)

(i) State sandwich theorem.

Ans If θ is measured in radian, then $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ is always equal to 1 which is called sandwich theorem.

(ii) Express the area "A" of a circle as a function of its circumference "C".

Ans Let 'r' be the radius of the circle A and C denote Area and circumference of the circle, then

$$A = \pi r^2 \qquad (1)$$

$$C = 2\pi r$$

$$\frac{C}{2\pi} = r \qquad (2)$$

Put value of 'r' in eq. (1),

$$A = \pi\left(\frac{C}{2\pi}\right)^2$$

$$= \pi\frac{C^2}{4\pi^2} = \frac{1}{4\pi}C^2$$

(iii) If $f(x) = \begin{cases} x+2 & , x \le -1 \\ c+2 & , x > -1 \end{cases}$, find "c" so that $\lim_{x \to -1} f(x)$ exists.

Ans We find left hand limit and right hand limit of f(x) at x = −1.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x+2) = -1 + 2 = 1$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (c+2) = c+2$$

If $\lim_{x \to -1} f(x)$ exists because

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$$

$$1 = c + 2$$

$$1 - 2 = c$$

$$\boxed{-1 = c}$$

(iv) Define differentiation.

Ans In an equation, if derivative of a dependent variable w.r.t independent variable exists. Then the process of finding the derivatives is called differentiation.

(v) Differentiate $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$ w.r.t x.

Ans Let

$$y=\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$$

$$=(\sqrt{x})^2+\left(\frac{1}{\sqrt{x}}\right)^2-2\sqrt{x}\cdot\frac{1}{\sqrt{x}}$$

$$y=x+\frac{1}{x}-2$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(x+\frac{1}{x}-2\right)$$

$$=\frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{1}{x}\right)-\frac{d}{dx}(2)$$

$$=1+\frac{x(0)-1(1)}{x^2}-0=1-\frac{1}{x^2}$$

$$\frac{dy}{dx}=\frac{x^2-1}{x^2}$$

(vi) Find $\frac{dy}{dx}$ if $xy+y^2=0$.

Ans

$$xy+y^2=0$$

$$x(y)'+y(x)'+(y^2)'=0$$

$$x\frac{dy}{dx}+y(1)+2y\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}+2y\frac{dy}{dx}=-y$$

$$(x+2y)\frac{dy}{dx}=-y$$

$$\frac{dy}{dx}=\frac{-y}{x+2y}$$

(vii) Find $\frac{dy}{dx}$ if $y=x\cos y$.

Ans

$$y=x\cos y$$

$$\frac{dy}{dx}=x(\cos y)'+\cos y\,(x)'$$

$$= x(-\sin y)\frac{dy}{dx} + \cos y\,(1)$$

$$= -x \sin y \frac{dy}{dx} + \cos y$$

$$\frac{dy}{dx} + x \sin y \frac{dy}{dx} = \cos y$$

$$(1 + x \sin y)\frac{dy}{dx} = \cos y$$

$$\frac{dy}{dx} = \frac{\cos y}{1 + x \sin y}$$

(viii) **Prove that $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$.**

Ans Let $y = \cos^{-1} x$

$\cos y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$-\sin y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{-\sin y}$$

$$= -\frac{1}{\sqrt{1 - \cos^2 y}}$$

$$= -\frac{1}{\sqrt{1 - x^2}} \quad \text{for } -1 < x < 1$$

(ix) **Find $\frac{dy}{dx}$ if $y = x e^{\sin x}$.**

Ans

$$y = x e^{\sin x}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x e^{\sin x})$$

$$= x\frac{d}{dx}(e^{\sin x}) + e^{\sin x}\frac{d}{dx}(x)$$

$$= x(e^{\sin x} \cos x) + e^{\sin x}(1)$$

$$= e^{\sin x}(x \cos x + 1)$$

(x) **Define power series.**

Ans A series of the form $a_0 + a_1x + a_2 x^2 + a_3 x^3 \ldots\ldots a_nx^n$ is called a power series of function f(x) where $a_0, a_1, a_2 \ldots\ldots a_n$ are constants and x is a variable.

(xi) **Find extreme values for $f(x) = x^2 - x - 2$.**

Ans

$$f'(x) = \frac{d}{dx}(x)^2 - \frac{d}{dx}(x) - \frac{d}{dx}(2)$$
$$= 2x - 1 - 0$$
$$f'(x) = 2x - 1$$
$$f''(x) = 2\frac{d}{dx}(x) - \frac{d}{dx}(1)$$
$$= 2(1) - 0$$
$$f''(x) = 2$$

Let $f'(x) = 0$

$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$
$$f''(x) = 2 > 0$$

so $f(x)$ has maximum value

at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$$
$$= \frac{1}{4} - \frac{1}{2} - 2$$
$$= \frac{1 - 2 - 8}{4} = \frac{-9}{4}$$

(xii) **Find $\frac{dy}{dx}$ if $y = \sin h^{-1}\left(\frac{x}{2}\right)$.**

Ans

$$y = \sin h^{-1}\left(\frac{x}{2}\right)$$
$$\sin h\, y = \frac{x}{2}$$

Differentiate both sides

$$\frac{d}{dx}(\sin h\, y) = \frac{d}{dx}\left(\frac{x}{2}\right)$$
$$\cos h\, y \frac{dy}{dx} = \frac{1}{2}$$
$$\sqrt{1 + \sin h^2 y}\,\frac{dy}{dx} = \frac{1}{2}$$
$$\sqrt{1 + \left(\frac{x}{2}\right)^2}\,\frac{dy}{dx} = \frac{1}{2}$$

$$\sqrt{1+\frac{x^2}{4}}\frac{dy}{dx}=\frac{1}{2}$$

$$\sqrt{\frac{4+x^2}{4}}\frac{dy}{dx}=\frac{1}{2}$$

$$\frac{\sqrt{4+x^2}}{2}\frac{dy}{dx}=\frac{1}{2}$$

$$\frac{dy}{dx}=\frac{1}{2}\times\frac{2}{\sqrt{4+x^2}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{4+x^2}}$$

3. Write short answers to any EIGHT (8) questions: (16)

(i) Find $\frac{dy}{dx}$ using differentials if $xy - \log_e x = c$.

Ans

$$xy - \log_e x = c$$

$$xy - \ln x = c$$

Differentiate w.r.t 'x'

$$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) - \frac{d}{dx}(\ln x) = \frac{d}{dx}(c)$$

$$x\frac{dy}{dx} + y(1) - \frac{1}{x} = 0$$

$$x\frac{dy}{dx} = \frac{1}{x} - y$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} - y}{x} = \frac{1-xy}{x^2}$$

(ii) Evaluate the integral $\int \frac{x}{x+2}\,.\,dx$.

Ans

$$\int \frac{x}{x+2}\,.\,dx$$

$$= \int \frac{x+2-2}{x+2}\,.\,dx$$

$$= \int \left(\frac{x+2}{x+2} - \frac{2}{x+2}\right) dx$$

$$= \int \left[1 - \left(\frac{2}{x+2}\right)\right] dx$$

$$= \int 1\, dx - 2\int \frac{1}{x+2} dx$$

$$= x - 2\, ln\,(x+2) + c$$

(iii) **Evaluate the integral $\int \frac{1}{a^2 - x^2} . dx$.**

Ans

$$\int \frac{1}{a^2 - x^2} . dx$$

$$= \frac{-1}{2x} \int \frac{-2x}{a^2 - x^2} dx$$

$$= \frac{-1}{2x} ln\, |a^2 - x^2| + c$$

(iv) **Evaluate the integral $\int x \sin x \cos x\, dx$.**

Ans

$$\int x \sin x \cos x\, dx$$

$$= \int x(\sin x \cos x)\, dx$$

Integration by parts

$$= x \frac{\sin^2 x}{2} - \int \frac{\sin^2 x}{2} (1)\, dx$$

$$= \frac{x}{2} \sin^2 x - \frac{1}{4} \int 2 \sin^2 x\, dx$$

$$= \frac{x}{2} \sin^2 x - \frac{1}{4} \int (1 - \cos 2x)\, dx$$

$$= \frac{x}{2} \sin^2 x - \frac{1}{4} \int 1\, dx + \frac{1}{4} \int \cos 2x\, dx$$

$$= \frac{x}{2} \sin^2 x - \frac{1}{4} x + \frac{1}{4} . \frac{\sin 2x}{2} + c$$

$$= \frac{x}{2} \sin^2 x - \frac{1}{4} x + \frac{1}{8} \sin 2x + c$$

(v) **Evaluate the integral $\int x^2 e^{ax} . dx$.**

Ans

$$= \int x^2 e^{ax} . dx$$

Integrating by parts

$$= x^2 \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} (2x)\, dx$$

$$= \frac{x^2}{a} e^{ax} - \frac{2}{a} \int e^{ax} . x\, dx$$

Again integrating by parts

$$= \frac{x^2}{a} e^{ax} - \frac{2}{a}\left[x \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}(1)\, dx\right]$$

$$= \frac{x^2}{a} e^{ax} - \frac{2}{a}\left(\frac{x\, e^{ax}}{a} - \frac{1 e^{ax}}{a^2}\right) + c$$

$$= \frac{x^2}{a} e^{ax} - \frac{2x\, e^{ax}}{a^2} + \frac{2e^{ax}}{a^3} + c$$

$$= \frac{1}{a} e^{ax}\left[x^2 - \frac{2}{a}x + \frac{2}{a^2}\right] + c$$

(vi) Evaluate the integral $\int e^{3x}\left(\frac{3 \sin x - \cos x}{\sin^2 x}\right) dx$.

Ans

$$= \int e^{3x}\left(\frac{3 \sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$$

$$= \int e^{3x}\left(\frac{3}{\sin x} - \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\right) dx$$

$$= \int e^{3x}(3 \operatorname{cosec} x - \operatorname{cosec} x \,.\, \cot x)\, dx$$

$$= 3\int e^{3x} \operatorname{cosec} x\, dx - \int e^{3x} \cot x \operatorname{cosec} x\, dx$$

Integrate the first integral by parts

$$= 3 \operatorname{cosec} x \,.\, \frac{e^{3x}}{3} - 3\int \frac{e^{3x}}{3}(-\cot x \operatorname{cosec} x)\, dx - \int e^{3x} \cot x \operatorname{cosec} x\, dx$$

$$= \operatorname{cosec} x \,.\, e^{3x} + 3\int \frac{e^{3x}}{3} \cot x \operatorname{cosec} x\, dx - \int e^{3x} \cot x \operatorname{cosec} x\, dx$$

$$= e^{3x} \operatorname{cosec} x + c$$

(vii) Prove that $\int_a^b f(x).dx = -\int_a^b f(x) \,.\, dx$.

Ans If $f(x) = \phi'(x)$

where ϕ is an anti-derivative of f

So by using fundamental theorem, we get

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a)$$

$$= -[\phi(a) - \phi(b)]$$

$$\int_a^b f(x)\, dx = -\int_a^b f(x) \,.\, dx.$$

(viii) Evaluate the definite integral $\int_0^3 \frac{dx}{x^2+9}$.

Ans

$$\int_0^3 \frac{dx}{x^2+9}$$

$$= \frac{1}{3}\left|\tan^{-1}\frac{x}{3}\right|_0^3$$

$$= \frac{1}{3}[\tan^{-1} 1 - \tan^{-1} 0]$$

$$= \frac{\pi}{12}$$

(ix) Find the area bounded by cos function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

Ans Let $y = \cos x$

so $\int_{-\pi/2}^{\pi/2} y\,dx$

$$= \int_{-\pi/2}^{\pi/2} \cos x\,dx$$

$$= |\sin x|_{-\pi/2}^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)$$

$$= 1 - (-1) = 1 + 1$$

$$= 2 \text{ Sq. units}$$

(x) Solve the differential equation $\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$.

Ans

$$\sin y \operatorname{cosec} x \frac{dy}{dx} = 1$$

$$\sin y \frac{1}{\sin x}\frac{dy}{dx} = 1$$

Separating the variables

$$\sin y\,dy = \sin x\,dx$$

Integrating both sides, we have

$$\int \sin y\,dy = \int \sin x\,dx$$

$$-\cos y = -\cos x + c$$

(xi) Define optimal solution and feasible solution.

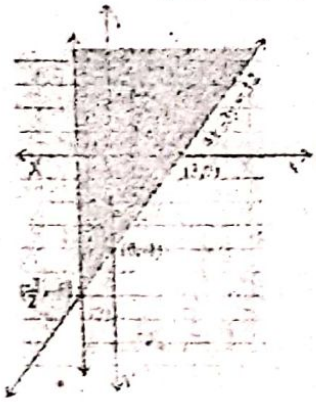
Ans The feasible solution which maximizes or minimizes the objective function is called the optimal solution. Each point of feasible region is called a feasible solution of the system of linear inequalities.

(xii) Graph the region indicated by $4x - 3y \leq 12$, $x \geq -\frac{3}{2}$.

Ans

$$4x - 3y \leq 12$$

Put $x = 0$, $y = 0$, we have $P(3, 0)(0, 4)$



As $\quad x \geq -\frac{3}{2} \quad ; \quad x = \frac{-3}{2}$

$$4\left(\frac{-3}{2}\right) - 3y = 12$$
$$2(-3) - 3y = 12$$
$$-6 - 3y = 12$$
$$-3y = 12 + 6$$
$$y = \frac{18}{-3}$$
$$y = -6$$
$$P(-\frac{3}{2}, -6)$$

4. Write short answers to any NINE (9) questions: (18)

(i) Show that the points A(3, 1), B(−2, −3) and C(2, 2) are vertices of an isosceles triangle.

Ans By using distance formula:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-2 - 3)^2 + (-3 - 1)^2}$$
$$= \sqrt{(-5)^2 + (-4)^2} = \sqrt{25 + 16} = \sqrt{41}$$

$$|BC| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(2+2)^2 + (2+3)^2}$$
$$= \sqrt{(4)^2 + (5)^2} = \sqrt{16+25} = \sqrt{41}$$
$$|CA| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(2-3)^2 + (2-1)^2}$$
$$= \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

So $|AB| = |BC|$

As two sides of triangle are equal.

So A, B and C are vertices of an isosceles triangle.

(ii) Find an equation of a line through the points (–2, 1) and (6, –4).

Ans
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

By putting values, we have

$$y - 1 = \frac{-4-1}{6-(-2)}(x - (-2))$$
$$y - 1 = \frac{-5}{6+2}(x+2) = \frac{-5}{8}(x+2)$$
$$8(y-1) = -5x - 10$$
$$8y - 8 = -5x - 10$$
$$5x + 8y - 8 + 10 = 0$$
$$5x + 8y + 2 = 0$$

(iii) Find an equation of the line bisecting the first and third quadrants.

Ans It passes through (0, 0) having slope 1. The equation of line bisecting the first and third quadrant is $y = x$.

(iv) Find an equation of the line with x-intercept: –3 and y – intercept : 4.

Ans
$$a = -3, \quad b = 4$$
$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{-3} + \frac{y}{4} = 1 \quad \Rightarrow \quad -4x + 3y = 12$$

(v) Convert $2x - 4y + 11 = 0$ into slope intercept form.

Ans $2x + 11 = 4y$

$$y = \frac{2x + 11}{4}$$

$$= \frac{2x}{4} + \frac{11}{4}$$

$$y = \frac{x}{2} + \frac{11}{4}$$

(vi) Write an equation of the parabola with focus (–1, 0), vertex (–1, 2).

Ans By using distance formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-1 + 1)^2 + (2 - 0)^2}$$

$$= \sqrt{(0)^2 + (2)^2} = \sqrt{(2)^2} = 2$$

As focus is below the vertex. So, equation of parabola is:

$$(x - h)^2 = -4a(y - k)$$

As vertex = (–1, 2) so h = –1, k = 2

$$[x - (-1)]^2 = -4(2)(y - 2)$$

$$(x + 1)^2 = -8(y - 2)$$

$$x^2 + 1 + 2x = -8y + 16$$

$$x^2 + 8y + 2x + 1 - 16 = 0$$

$$x^2 + 2x + 8y - 15 = 0$$

(vii) Find the focus and directrix of the parabola $y = 6x^2 - 1$.

Ans

$$y = 6x^2 - 1$$

$$6x^2 = y + 1$$

$$x^2 = \frac{1}{6}(y + 1)$$

Shift the origin to (0, –1)

Let x = X and y + 1 = Y

$$X^2 = \frac{1}{6}Y \qquad \text{(i)}$$

Compare equation (i) with $X^2 = 4aY$

$$4aY = \frac{1}{6}Y$$

$$4a = \frac{1}{6}$$

$$a = \frac{1}{24}$$

As vertex (0, 0), so X = 0; Y = 0

$$x = 0,\ y + 1 = 0$$
$$y = -1 \quad \Rightarrow \quad v(0, -1)$$

So focus is (0, a)

$$= \left(0, \frac{1}{24}\right)$$

Here $X = 0, \quad Y = \frac{1}{24}$

$$x = 0,\ y + 1 = \frac{1}{24}$$
$$y = \frac{1}{24} - 1$$
$$= \frac{23}{24}$$
$$F(0, \frac{23}{24})$$

Directrix: $Y = -a$

$$y + 1 = \frac{-1}{24}$$
$$= \frac{-1}{24} - 1$$
$$= \frac{-1 - 24}{24}$$
$$y = \frac{-25}{24}$$
$$24y + 25 = 0$$

(viii) Find an equation of the ellipse with centre (0, 0), focus (0, –3), vertex (0, 4).

Ans As $f(0, -c)$ so $c = 3$

$V(0, a)$ so $a = 4$

$$c^2 = a^2 + b^2$$
$$b^2 = a^2 - c^2$$
$$b^2 = (4)^2 - (3)^2 = 16 - 9$$
$$b^2 = 7$$

Thus equation of ellipse is

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$
$$\frac{y^2}{16} + \frac{x^2}{7} = 1$$

(ix) Find the eccentricity and directrices of the ellipse whose equation is $25x^2 + 9y^2 = 225$.

Ans

$$\frac{25}{225}x^2 + \frac{9}{225}y^2 = 1$$

$$\frac{1}{9}x^2 + \frac{1}{25}y^2 = 1$$

Here $a^2 = 25 \;;\; b^2 = 9$

$a = 5 \;;\; b = 3$

$c^2 = a^2 + b^2$

$= 25 - 9 = 16$

$c^2 = 16$

$c = \pm 4$

Foci are $(0, \pm c) = (0, \pm 4)$

Vertices are $(0, \pm a) = (0, \pm 5)$

$$c = a\,e \quad \Rightarrow \quad e = \frac{c}{a} = \frac{4}{5}$$

$$\text{Directrices} = y = \pm\frac{a}{e}$$

$$= \pm\frac{5}{\frac{4}{5}} = \pm\frac{25}{4}$$

(x) Define unit vector.

Ans A unit vector is defined as a vector whose magnitude is unity. It is written as $\hat{\underline{v}}$ and is defined by $\hat{\underline{v}} = \frac{v}{|v|}$.

(xi) Find a unit vector in the direction of the vector $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$.

Ans

$$\underline{v} = \frac{v}{|V|}$$

$$|v| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{\left(\frac{2}{2}\right)^2} = 1$$

$$\underline{v} = \frac{v}{|V|} = \frac{\frac{1}{2}i + \frac{\sqrt{3}}{2}j}{1}$$

$$v = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$$

(xii) Find a vector whose magnitude is '4' and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$.

Ans Let

$$\underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$$

$$\therefore \quad |\underline{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2}$$
$$= \sqrt{4 + 9 + 36}$$
$$= \sqrt{49}$$
$$= 7$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}$$

Thus the required vector is:

$$\hat{\underline{v}} = 4\left(\frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}\right)$$

$$\hat{\underline{v}} = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

(xiii) Find a scalar "α" so that the vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

Ans Let $\underline{v} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$.

$\underline{u} = 3\underline{i} + \underline{j} + \alpha\underline{k}$

$\underline{v} \perp$ to $\underline{u}$.

$\underline{v} \cdot \underline{u} = 0$

$(2\underline{i} + \alpha\underline{j} + 5\underline{k}0) \cdot (3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$

$(2)(3) + (\alpha)(1) + (5)(\alpha) = 0$

$6 + 6\alpha = 0$

$6\alpha = -6$

$\alpha = -\frac{6}{6}$

$\alpha = -1$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, & x \neq 2 \\ k & x = 2 \end{cases}$ find the value of k, so that 'f' is continuous at $x = 2$. (5)

(xii) Find a vector whose magnitude is '4' and is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$.

Ans Let

$$\underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$$

$$\therefore \quad |\underline{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2}$$
$$= \sqrt{4 + 9 + 36}$$
$$= \sqrt{49}$$
$$= 7$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}$$

Thus the required vector is:

$$\hat{\underline{v}} = 4\left(\frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}\right)$$

$$\hat{\underline{v}} = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

(xiii) Find a scalar "α" so that the vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

Ans Let $\underline{v} = 2\underline{i} + \alpha\underline{j} + 5\underline{k}$.

$$\underline{u} = 3\underline{i} + \underline{j} + \alpha\underline{k}$$

$\underline{v} \perp$ to $\underline{u}$.

$$\underline{v} \,.\, \underline{u} \,. = 0$$

$$(2\underline{i} + \alpha\underline{j} + 5\underline{k}0) \,.\, (3\underline{i} + \underline{j} + \alpha\underline{k}) = 0$$

$$(2)(3) + (\alpha)(1) + (5)(\alpha) = 0$$

$$6 + 6\alpha = 0$$

$$6\alpha = -6$$

$$\alpha = -\frac{6}{6}$$

$$\alpha = -1$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, & x \neq 2 \\ k & x = 2 \end{cases}$ find the value of k, so that 'f' is continuous at x = 2. (5)

Ans
$$f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & x \neq 2 \\ k & x = 2 \end{cases}$$

$f(2) = k$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$$

$$= \lim_{x \to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{(x-2)} \times \frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$= \lim_{x \to 2} \frac{(\sqrt{2x+5})^2-(\sqrt{x+7})^2}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$= \lim_{x \to 2} \frac{2x+5-x-7}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$= \lim_{x \to 2} \frac{(x-2)}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$= \lim_{x \to 2} \frac{1}{(\sqrt{2x+5}+\sqrt{x+7})} = \frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}}$$

$$= \frac{1}{3+3}$$

$$\lim_{x \to 2} f(x) = \frac{1}{6}$$

'f' is continuous, so

$$f(2) = \lim_{x \to 2} f(x)$$

$$k = \frac{1}{6}$$

(b) Show that $y = x^x$ has maximum value at $x = \frac{1}{e}$. **(5)**

Ans $y = x^x$

Taking 'log' on both sides,

$$\ln y = \ln (x^x)$$
$$= x \ln x$$

Differentiating w.r.t, 'x'

$$\frac{d}{dx}(lx\ y) = \frac{d}{dx}(x \,.\, \ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \,.\, \frac{1}{x} + \ln x\ (1)$$
$$= 1 + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$
$$= x^x(1 + \ln x)$$

If $\frac{dy}{dx} = 0$

$$x^x(1 + \ln x) = 0$$
$$1 + \ln x = 0$$
$$\ln e + \ln x = 0 \qquad \because (1 = \ln e)$$
$$\ln(e \,.\, x) = 0$$
$$\ln(e \,.\, x) = \ln 1 \qquad (\because 0 = \ln 1)$$
$$e \,.\, x = 1$$
$$x = \frac{1}{e}$$

Again differentiating w.r.t 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(x^x(1 + \ln x))$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^x) \,.\, (1 + \ln x) + x^x \frac{d}{dx}(1 + \ln x)$$
$$= x^x(1 + \ln x)(1 + \ln x) + x^x\left(0 + \frac{1}{x}\right)$$
$$= x^x(1 + \ln x)^2 + x^x\left(\frac{1}{x}\right)$$
$$= x^x\left[(1 + \ln x)^2 + \frac{1}{x}\right]$$

As $x = \frac{1}{e}$, so

$$= \left(\frac{1}{e}\right)^{1/e}\left[\left(1 + \ln\frac{1}{e}\right)^2 + \frac{1}{1/e}\right]$$
$$= \left(\frac{1}{e}\right)^{1/e}[(1 + \ln 1 - \ln e)^2 + e]$$
$$= \left(\frac{1}{e}\right)^{1/e}[(1 + 0 - 1)^2 + e]$$
$$= \left(\frac{1}{e}\right)^{1/e}[0 + e]$$
$$= \left(\frac{1}{e}\right)^{1/e}(e) <> 0$$

So y has maximum value at $x = \frac{1}{e}$.

Q.6.(a) Evaluate $\int e^{2x} \cos 3x \, dx$. (5)

Ans Let $I = \int e^{2x} \cos 3x \, dx$

Integrating by parts,

$$I = e^{2x} \cdot \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} \cdot e^{2x} (2) \, dx$$

$$= \frac{\sin 3x}{3} e^{2x} - \frac{2}{3} \int \sin 3x \cdot e^{2x} \, dx$$

$$= e^{2x} \frac{\sin 3x}{3} - \frac{2}{3} \int e^{2x} \sin 3x \, dx$$

$$= e^{2x} \frac{\sin 3x}{3} - \frac{2}{3} \int \left[e^{2x} \cdot \frac{-\cos 3x}{3} - \int \left(\frac{-\cos 3x}{3} \right) e^{2x} (2) \, dx \right.$$

$$= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + \frac{2}{3} \int \frac{-\cos 3x}{3} \cdot e^{2x} (2) \, dx$$

$$= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x \, dx$$

$$I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} I + c$$

$$I + \frac{4}{9} I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c$$

$$I \left(1 + \frac{4}{9} \right) = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c$$

$$I \left(\frac{9+4}{9} \right) = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c$$

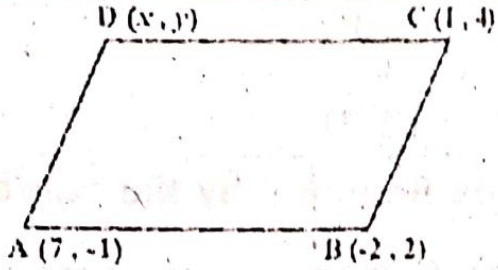
$$I \left(\frac{13}{9} \right) = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x + c$$

$$I = \frac{9}{13} \left(\frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x \right) + \frac{9}{13} c$$

(b) The three points A(7, –1), B(–2, 2) and C(1, 4) are consecutive vertices of a parallelogram, find the fourth vertex. (5)

Ans Let (x, y) be the coordinates of vertex D.

As ABCD is a parallelogram. So, AB = CD and the slope of side BC = DA.



$$\text{Slope } AB = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2-(-1)}{-2-7} = \frac{2+1}{-2-7} = \frac{3}{-9} = -\frac{1}{3}$$

$$\text{Slope } BC = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4-2}{1-(-2)} = \frac{2}{3}$$

$$\text{Slope } CD = \frac{y-4}{x-1}$$

$$\text{Slope } DA = \frac{-1-y}{7-x}$$

As Slope AB = CD

$$\frac{-1}{3} = \frac{y-4}{x-1}$$

$$-1(x-1) = 3(y-4)$$

$$-1x + 1 = 3y - 12$$

$$-1x - 3y = -12 - 1$$

$$+1x - 3y = +13 \qquad \text{(i)}$$

As Slope BC = DA

$$\frac{2}{3} = \frac{-1-y}{7-x}$$

$$2(7-x) = 3(-1-y)$$

$$14 - 2x = -3 - 3y$$

$$14 + 3 = 2x - 3y$$

$$17 = 2x - 3y \qquad \text{(ii)}$$

Adding equations (i) and (ii), we get

$$3x = 30$$

$$x = 10$$

Put $x = 10$ in eq.(ii),

$$17 = 2(10) - 3(y)$$

$$17 = 20 - 3y$$

$$17 - 20 = -3y$$

$$-3 = -3y$$
$$y = 1$$

Thus vertex D = (10, 1).

Q.7.(a) Find the area bounded by the curve $y = x^3 - 4x$ and x-axis. (5)

Ans

$$y = x^3 - 4x$$

Let
$$y = 0$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0$$

$x = 0 \quad ; \quad x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = 2, -2$

$$x = (0, \pm 2)$$

$$\text{Area} = \int_a^b y\,dx$$
$$= \int_a^b (x^3 - 4x)\,dx$$
$$= \int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{+2} (x^3 - 4x)\,dx$$
$$= \int_{-2}^{0} x^3\,dx - 4\int_{-2}^{0} x\,dx + \int_{0}^{+2} x^3\,dx - 4\int_{0}^{+2} x\,dx$$
$$= \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{-2}^{0} - \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{0}^{+2}$$
$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{+2}$$
$$= 0 - \left[\frac{(-2)^4}{4} - 2(-2)^2\right] - \left[\frac{(2)^4}{4} - 2(2)^2\right] - 0$$
$$= -\left[\frac{16}{4} - 8\right] - \left[\frac{16}{4} - 8\right]$$
$$= -[4 - 8] - [4 - 8]$$
$$= -[-4] - [-4]$$
$$= 4 + 4 = 8 \text{ sq. units.}$$

(b) Minimize $z = zx + y$ subject to the constraints: **(5)**

$x + y \geq 3,\ 7x + 5y \leq 35,\ x \geq 0,\ y \leq 0$

Ans Equation

$x + y = 3$

Put $y = 0$

$x = 3$

$(3, 0)$

Put $x = 0$

$y = 3$

$(0, 3)$

$0 + 0 \geq 3$

$0 \ngeq 3$

$7x + 5y = 35$

Put $y = 0$

$x = \dfrac{35}{7}$

$x = 5$

$(5, 0)$

Put $x = 0$

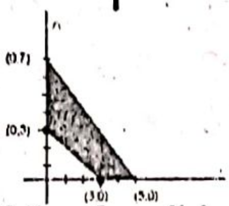
$5y = 35$

$y = 7$

$(0, 7)$

$0 + 0 \leq 35$

$0 \leq 35$



The corner points of the feasible region (3, 0), (5, 0) (0, 7) and (0, 3)

$Z = 2x + y$

$Z = 2(3) + 0$

$(3, 0)$

$= 6$

$Z(5, 0) = 2(5) + 0$

$= 10$

$Z(0, 7) = 2(0) + 7 = 7$

$Z(0, 3) = 2(0) + 3 = 3$

'Z' is minimize at the corner point (0, 3).

Q.8.(a) Find the condition that the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$ at a single point. **(5)**

Ans

$y = mx + c$

Put it in $x^2 + y^2 = a^2$

$x^2 + (mx + c)^2 = a^2$

$x^2 + m^2x^2 + c^2 + 2mcx = a^2$

$x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0 \qquad \text{(i)}$

Disc '$b^2 - 4ac$' of equation (i),

$= (2mc)^2 - 4(1 + m^2)(c^2 - a^2)$

$$= 4m^2c^2 - 4c^2(1 + m^2) + 4a^2(1 + m^2)$$
$$= 4m^2c^2 - 4c^2 - 4m^2c^2 + 4a^2 + 4m^2a^2$$
$$= -4c^2 + 4a^2 + 4m^2a^2$$
$$= 4[-c^2 + a^2(1 + m^2)]$$

For $y = mx + c$ touches the circle, so

$$4(-c^2 + a^2(1 + m^2)) = 0$$
$$-c^2 + a^2(1 + m^2) = 0$$
$$c^2 = a^2(1 + m^2) \text{ Required condition.}$$

(b) Find x so that points A(1, –1, 0), B(–2, 2, 1) and C(0, 2, x) form triangle with right angle at C. (5)

Ans

$$\vec{AC} = (0, 2, x) - (1, -1, 0)$$
$$= 0 - 1, 2 + 1, x - 0$$
$$\overline{AC} = -1\hat{i} + 3\hat{j} + x\hat{k}$$
$$\overline{BC} = (0, 2, x) - (-2, 2, 1)$$
$$= 0 + 2, 2 - 2, x - 1$$
$$= +2\hat{i} + 0\hat{j} + (x - 1)\hat{k}$$

Given $AC \perp BC$ So $AC \cdot BC = 0$

$$\Rightarrow (-i + 3j + x k) \cdot (2\hat{i} + 0\hat{j} + (x - 1)\hat{k}) = 0$$
$$\Rightarrow (-1)(2) + 3(0) + x(x - 1) = 0$$
$$x^2 - x - 2 = 0$$
$$x^2 - 2x + x - 2 = 0 \qquad (x - 2)(x + 1)$$
$$x(x - 2) + 1(x - 2) = 0 \qquad x = 2, -1$$

Q.9.(a) Find the centre, foci, eccentricity, vertices and equations of directices of $\frac{y^2}{4} - x^2 = 1$. (5)

Ans

$$\frac{y^2}{4} - x^2 = 1 \qquad \text{(i)}$$

As we know

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

So equation (i) becomes

$$\frac{(y - 0)^2}{(2)^2} - \frac{(x - 0)^2}{(1)^2} = 1$$

Here $a = 2, \; b = 1$

$$\boxed{\text{centre} = (0, 0)}$$

$$c^2 = a^2 + b^2$$
$$= (2)^2 + (1)^2 = 4 + 1 = 5$$
$$c = \sqrt{5}$$

$$\text{Eccentricity} = e = \frac{c}{a} \Rightarrow \boxed{e = \frac{\sqrt{5}}{2}}$$

$$\text{Focus} = (0, \pm c)$$
$$\boxed{F = 0, \pm\sqrt{5}}$$
$$V = (0, \pm a)$$
$$\boxed{V = (0, \pm 2)}$$

$$\text{Equation of directrices } y = \pm\frac{a}{e} = \pm\frac{2}{\frac{\sqrt{5}}{2}}$$

$$y = \pm\frac{4}{\sqrt{5}}$$

(b) Find volume of the tetrahedron with the vertices A(2, 1, 8), B(3, 2, 9), C(2, 1, 4) and D(3, 3, 10). (5)

Ans
$$\overrightarrow{AB} = (3-2)\,\underline{i} + (2-1)\,\underline{j} + (9-8)\,\underline{k} = \underline{i} + \underline{j} + \underline{k}$$
$$\overrightarrow{AC} = (2-2)\,\underline{i} + (1-1)\,\underline{j} + (4-8)\,\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$$
$$\overrightarrow{AD} = (3-2)\,\underline{i} + (3-1)\,\underline{j} + (10-8)\,\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$$

$\therefore$ Volume of the tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$
$$= \frac{1}{6}[1(0+8) - 1(0+4) + 1(0, 0)]$$
$$= \frac{1}{6}[8-4]$$
$$= \frac{1}{6}[4(2-1)]$$
$$= \frac{4}{6} = \frac{2}{3}$$